

# Quantum Hall - insulator transitions in lattice models with strong disorder

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## Abstract

We report results of numerical studies of the integer quantum Hall effect in a tight binding model on a two-dimensional square lattice with non-interacting electrons, in the presence of a random potential as well as a uniform magnetic field applied perpendicular to the lattice. We consider field magnitudes such that the area per flux quantum is commensurate with the lattice structure. Topological properties of the single electron wave functions are used to identify current carrying states that are responsible for the quantized Hall conductance. We study the interplay between the magnetic field and the disorder, and find a universal pattern with which the current carrying states are destroyed by increasing disorder strength, and the system driven into an insulating state. We also discuss how to interpolate results of lattice models to the continuum limit. The relationship to previous theoretical and experimental studies of quantum Hall-insulator transitions in strongly disordered

systems at low magnetic fields is discussed.

## I. INTRODUCTION

The integer quantum Hall effect (IQHE) is observed in two-dimensional (2D) electron systems, in the presence of a strong perpendicular magnetic field.<sup>1</sup> Current understanding of IQHE is mostly based on studies of non-interacting electron models, with the simplifying assumption that electron-electron interaction does not play an important role. Indeed, the phenomenology of IQHE may be explained very well by the non-interacting electron model in a strong magnetic field: the magnetic field gives rise to Landau quantization and the large gap necessary for dissipationless transport, while disorder localizes electronic states away from the centers of Landau bands, such that the Hall conductance remains quantized as the Fermi energy lies in the localized region of the density of states (DOS); transitions between different quantum Hall phases occur when the Fermi energy sweeps through critical energies in the middle of Landau bands, where states are delocalized (i.e., have a diverging localization length).<sup>2,3</sup>

The scaling theory of localization<sup>4</sup> predicts that all states in a system of noninteracting electrons are localized in 2D, in the absence of a magnetic field. Therefore a 2D electron gas system must undergo a phase transition (or a series of phase transitions) from a quantum Hall state to the insulating phase, as the strength of the magnetic field ( $B$ ) goes to zero.<sup>5</sup> The nature of such transition(s) is of current interest,<sup>6-9</sup> and is the subject of the present paper. On the theoretical front, Khmelnitskii<sup>10</sup> and Laughlin<sup>11</sup> proposed, based on semiclassical considerations, that delocalized states at centers of Landau bands “float” upward as  $B$  decreases and mixing between different Landau bands becomes important, and disappear at infinite energy in the  $B \rightarrow 0$  limit, thus reconciling the very different behavior at zero and strong magnetic fields. This simple and elegant picture is the basis of the global phase diagram of quantum Hall effect proposed by Kivelson, Lee and Zhang,<sup>12</sup> and has received support from both experimental<sup>13-16</sup> and microscopic theoretical<sup>17-22</sup> studies. On the other hand, based on numerical studies of a one-band tight binding model on a square lattice, authors of Refs. 23,24 (see also Refs. 25,26) concluded that instead of floating to infinite

energy in the  $B \rightarrow 0$  limit, the extended states disappear at finite  $B$ , and proposed a very different phase diagram. On the experimental side, the situation is also rather unclear. Although most experiments are consistent with the global phase diagram, people have routinely observed direct transition from quantum Hall phases with  $\nu > 1$  to an insulator as  $B$  decreases, where  $\nu$  is the Hall conductance in unit of  $\frac{e^2}{h}$ . This is in apparent contradiction with the global phase diagram, which predicts that  $\nu$  can only change by  $\pm 1$  at each integer quantum Hall transition. Some of these discrepancies, like the  $2 \rightarrow 0$  transitions,<sup>6,7</sup> may be explained as due to the small Zeeman splitting at low  $B$ , which gives rise to spin-unresolved quantum Hall transitions. The recently observed  $3 \rightarrow 0$  transition,<sup>9</sup> however, is apparently quite difficult to interpret along the same line.

Most of the previous theoretical studies focus on the continuum model. It was recently argued,<sup>23–26</sup> however, that the existence of an underlying lattice (and the corresponding conduction band) is important to understand the transition between a quantum Hall state and the insulating state at low  $B$ , even though the band width is much bigger than the Fermi energy and Landau level spacings. A crucial difference between the continuum model and tight-binding models is that in the latter case, there exist current carrying states that carry negative Hall conductance near the band center (and the total Hall conductance is zero for the entire band), while such states do not exist in the former case. However it remains unclear how to interpolate between the lattice and continuum models.

In this paper we present new results of numerical studies of the tight binding lattice model in the presence of a uniform magnetic field and random potential. Unlike previous studies<sup>20,23–28</sup> which have been focusing on special cases where there is particle-hole symmetry and critical energies carrying negative Hall conductance only exist in the central band(s), we have studied the most general case. We find, however, that the pattern by which the extended states disappear as disorder strength increases is rather similar to that of the special case studied earlier. We also explicitly discuss how to interpolate the numerical results on a lattice to the continuum limit properly.

The rest of the paper is organized in the following way. In section II we introduce the

model we study and the numerical approach we use to identify extended current carrying states, namely the calculation of a topological quantum number called Chern number. In section III we present our numerical results for various cases that we have studied. In section IV we summarize our results, and discuss their relations with previous theoretical and experimental studies.

## II. MODEL AND CHERN NUMBERS

We study numerically non-interacting electrons moving on a lattice with a uniform magnetic field and random onsite potential, described by the following tight binding Hamiltonian:

$$H = - \sum_{ij} t_{ij} (e^{ia_{ij}} c_i^\dagger c_j + e^{-ia_{ij}} c_j^\dagger c_i) + \sum_i \epsilon_i c_i^\dagger c_i, \quad (1)$$

where  $c_i$  is the fermion operator on lattice site  $i$ . The first term represents hopping or kinetic energy of the electrons, and the hopping phases  $a_{ij}$  are determined by the uniform magnetic field (up to arbitrary gauge choices). We consider the case of  $q/p$  flux quantum per square, where  $q$  and  $p$  are primitive integers. In this case, the original tight binding band (for zero field) is split into  $p$  Landau subbands in the absence of the random potential. The second term represents a random onsite potential. We take  $\epsilon_i$  to be uniformly distributed between  $-w$  and  $w$ , measured in units of the magnitude of the nearest neighbor hopping matrix element, which is set to be 1 throughout this paper.

The special case of the model (1) with near neighbor hopping only and  $q = 1$  has been studied in some detail before.<sup>20,23–28</sup> It is found in this case that for weak randomness, the extended states in most Landau subbands form a critical energy with total Chern number<sup>29</sup> (TCN)  $+1$ , while those in the central subband (odd  $p$ ) or two central subbands (even  $p$ ) split into two critical energies with negative TCNs. The total Chern number of the entire band is zero, as the filled band has zero Hall conductance. Increasing randomness strength merges critical energies with TCN  $+1$  and those with negative TCNs, with those closer to the band center disappearing at lower randomness.<sup>27,23,25</sup> Simultaneously, it is found that

the critical energies near the band edge float away from the centers of their corresponding Landau subbands towards the center of the whole band,<sup>20</sup> consistent with the “floating up” picture<sup>10,11</sup> in continuum systems where there is no critical energy with negative TCN. Such behavior is understood heuristically as due to an “attraction” between states or critical energies with Chern numbers of opposite signs.<sup>20</sup>

This special case discussed above has the very unique feature that in the absence of randomness all subbands carry Chern number +1 except for the central subband(s). In experiments however, the field strength is varied continuously. Also restricting the model to near neighbor hopping only endows the model with particle-hole symmetry, which is also absent in real systems<sup>3</sup>. For general field strength commensurate with the lattice, i.e.,  $q/p$  flux quantum per plaquette, the Chern numbers of subbands follow some nontrivial pattern and subbands carrying Chern numbers of opposite signs intertwine with each other. It is unclear how the extended states behave upon introducing randomness in this case. We address this issue numerically in the present paper, and without losing generality, we assume  $q/p < 1/2$ .

To study the localization property of this model, we use the Chern number approach,<sup>30–33</sup> which is briefly reviewed below. The Hall conductance of an individual eigenstate  $|m\rangle$  can be obtained easily using the Kubo formula:

$$\sigma_{xy}^m = \frac{ie^2\hbar}{A} \sum_{n \neq m} \frac{\langle m|v_y|n\rangle\langle n|v_x|m\rangle - \langle m|v_x|n\rangle\langle n|v_y|m\rangle}{(E_n - E_m)^2},$$

where  $A$  is the area of the system,  $v_x$  and  $v_y$  are the velocity operators in the  $x$  and  $y$  directions respectively. For a finite system with the geometry of a parallelogram with periodic boundary conditions (torus geometry),  $\sigma_{xy}^m$  depends on the two boundary condition phases  $\phi_1$  and  $\phi_2$ . As shown by Niu *et al.*, the boundary condition averaged Hall conductance takes the form<sup>31</sup>

$$\langle \sigma_{xy}^m \rangle = \frac{1}{4\pi^2} \int d\phi_1 d\phi_2 \sigma_{xy}^m(\phi_1, \phi_2) = C(m)e^2/h, \quad (2)$$

where  $C(m)$  is an integer called the Chern number of the state  $|m\rangle$ . States with nonzero

Chern numbers carry Hall current and are necessarily extended states<sup>32,33</sup>. Thus by numerically diagonalizing the Hamiltonian on a grid of  $\phi_1$  and  $\phi_2$ , and calculating the Chern numbers by converting the integral in (2) to a sum over grid points, we are able to identify extended states unambiguously. Depending on system size, the number of grids used in numerical calculations range from  $25 \times 25$  to  $50 \times 50$ .

### III. NUMERICAL RESULTS

In this section we present our numerical results. We start by discussing the case when there is only nearest neighbor hopping, and no disorder. In this case we have an exact particle-hole symmetry, and the quantized Hall conductance of each of the  $p$  Landau subbands (which is also the Chern number of this subband) is known exactly. For the  $t$ th subband, the quantized Hall conductance (in unit of  $e^2/h$ )  $\sigma_t$  when it is occupied *together with all the subbands below it*, satisfies the Diophantine equation<sup>34,30</sup>

$$t = \sigma_t q + sp, \quad (3)$$

where  $s$  is an integer that gives rise to the integer  $\sigma_t$  with the smallest magnitude satisfying Eq. (3).<sup>30,35</sup> The quantized Hall conductance of the  $t$ th subband itself, or its Chern number, is therefore

$$C_t = \sigma_t - \sigma_{t-1}. \quad (4)$$

In the following we show that the solutions of Eq. (3) (together with the condition below it) has a very non-trivial pattern in it. We start by discussing the case  $t = mq$ , where  $m$  is an integer. In this case Eq. (3) reduces to

$$\sigma_{mq} q = mq - sp. \quad (5)$$

For  $m < p/2$ , the solution with  $\sigma_{mq}$  having smallest magnitude is realized by choosing  $s = 0$ , and

$$\sigma_{mq} = m. \quad (6)$$

Therefore for  $m < p/2$ ,

$$\sum_{i=(m-1)q+1}^{mq} C_i = \sigma_{mq} - \sigma_{(m-1)q} = 1, \quad (7)$$

implying that in the lower half of the entire band, *subbands form groups of  $q$  subbands with total Chern number 1*. From particle-hole symmetry we know that the same happens in the upper half of the entire band as well; and the remaining  $M = \text{mod}(p, 2q)$  subbands at the center of the entire band therefore form a special group carrying negative total Chern numbers, so that the total Chern number for the entire band is zero.

The Chern numbers of individual subbands, however, depends sensitively on  $q$  and  $p$ , and oscillates between positive and negative integers. Therefore in the absence of disorder, the quantized Hall conductance does not have a simple monotonous dependence on the filling factor, even if the Fermi energy is far away from the band center. Also in this case it is not clear how current carrying states move and merge upon introducing disorder, based on the heuristic argument that “states carrying Chern numbers with opposite signs attract each other”.<sup>20</sup>

In the following we show that as randomness is introduced, it first merges the current carrying states *in the same group*. More specifically, current carrying states in groups away from the band center merge together and form critical energies carrying total Chern number +1, while those in the group at the band center form two critical energies carrying equal amount of negative Chern numbers. This is because the energy gaps separating subbands within the same group are much smaller than those separating neighboring groups, therefore the density of states as well as current carrying states in the same group merge together first. At this point the configuration of current carrying critical energies become the same as that of the cases with  $q = 1$ : There are critical energies with TCN +1 away from the band center, and two critical energies carrying negative TCN near the band center. Further increasing the randomness strength will merge the remaining critical energies, with the ones closer to the band center disappearing earlier.<sup>23–26</sup>

In the next two subsections we present numerical results for cases we have studied, to demonstrate the above observations. It should be borne in mind that the numerical results presented are for finite sized lattices, which may differ somewhat from the results of lattices in the thermodynamic limit (e.g., some critical energies which may be distinct in the latter case may not be resolved for our sizes). However, we wish to emphasize that we have taken care to make sure that all the relevant splittings are observable for our sizes, except possibly a small splitting of the central band in the particle-hole symmetric case.

### A. Cases with nearest neighbor hopping only

We first present our results for cases where there is nearest neighbor hopping only. In these cases we have particle-hole symmetry even in the presence of random potential, *after* physical quantities are averaged over all possible disorder configurations. The number of different disorder configurations (or “samples”) we explore numerically range from 200 to 800. The special case of  $q/p = 2/5$  has already been studied by Tan,<sup>28</sup> the results obtained by him agree with our general observations above.

$q/p = 3/7$ . In this case we have 7 subbands in the absence of disorder, and the Chern numbers of the subbands are  $(-2, 5, -2), (-2), (-2, 5, -2)$ , obtained from solving Eq. (3) with the condition below it, with ones in the same parentheses belong to the same group. In Fig.1 we show how the disorder averaged total density of states ( $\rho$ ), as well as density of current carrying states ( $\rho_c$ ), evolve as the randomness strength  $w$  increases, for systems of square geometry and size  $7 \times 7$ . We see for weak randomness of  $w = 0.5$ , the tails of  $\rho$  of different subbands in the same group have just started to overlap, and there are still well defined peaks of  $\rho_c$  for each individual subband. In the thermodynamic limit, current carrying extended states are expected to exist only at individual critical energies;<sup>2</sup> in finite size systems however,  $\rho_c$  has finite width around these critical energies as states with localization length bigger than the system size appear extended. As  $w$  is increased to 1.0, subbands in the same group have merged together, so have  $\rho_c$ , while there are still gaps

separating different groups. Further increasing  $w$  to 2.5,  $\rho$  of the 3 different group start to merge; the current carrying states in the central group (which consists of only a single subband in this case) split into two peaks (that become sharp critical energies each carrying TCN -1 in the thermodynamic limit), moving toward the band edges to meet the current carrying states in the side groups, which form two critical energies each carrying TCN +1 in the thermodynamic limit. At this point, as far as the behavior of current carrying states is concerned, the situation is identical to that of the case with  $q/p = 1/3$  which has been studied closely earlier,<sup>20,37</sup> even though initially we have 7 subbands in this case. Further increasing  $w$  to 4.0, the critical energies carrying opposite TCN have merged and disappeared, and there are no current carrying states anywhere in the thermodynamic limit; the system is in an insulating state no matter where the Fermi energy is. In a finite size system we see very low  $\rho_c$  which is expected to go to zero rapidly as system size increases.

$q/p = 2/7$ . In this case we again have 7 subbands in the absence of disorder, and the Chern numbers of the subbands are  $(-3, 4), (-3, 4, -3), (4, -3)$ , again with the ones in the same parentheses belong to the same group. We again study systems with size  $7 \times 7$ , and show in Fig.2 how  $\rho$  and  $\rho_c$  evolve with  $w$  in this case. For  $w = 0.5$ , the subbands in the two side groups have already merged together, while the tails of  $\rho$  of the central group (with 3 subbands) barely start to overlap. Increasing  $w$  to 1.0, the 3 subbands in the central group now have significant overlap, while the gaps separating different groups remain. As in the previous case, further increasing  $w$  to 2.5,  $\rho$  of the 3 different group start to merge, and the current carrying states in the central group split into two peaks, moving toward the band edges to meet the current carrying states in the side groups. Further increasing  $w$  to 4.0, all critical energies have merged and disappeared, and we see a very low  $\rho_c$ .

$q/p = 3/8$ . In this case we have 8 subbands in the absence of disorder, and the Chern numbers of the subbands are  $(3, -5, 3), (-1, -1), (3, -5, 3)$ .<sup>36</sup> We have studied systems with size  $8 \times 8$ . The situation is quite similar to the previous two cases. For  $w = 0.3$ ,  $\rho$  of the subbands in the same group have just started to overlap. As  $w$  is increased to 1.0, subbands in the same group have merged together, so have  $\rho_c$ , while there are still gaps separating

different groups. Further increasing  $w$  to 2.2,  $\rho$  of the 3 different group start to merge; the current carrying states in the central group split into two peaks, moving toward the band edges to meet the current carrying states in the side groups, which form two critical energies each carrying TCN +1 in the thermodynamic limit. Further increasing  $w$  to 3.0, the critical energies carrying opposite TCN have just merged and disappeared.

$q/p = 2/9$ . In this case we have 9 subbands in the absence of disorder, and the Chern numbers of the subbands are  $(-4, 5), (-4, 5), (-4), (5, -4), (5, -4)$ . The dependence of  $\rho$  and  $\rho_c$  on  $w$  for systems of size  $9 \times 9$  are shown in Fig. 4. For  $w = 0.5$ ,  $\rho$  and  $\rho_c$  of groups 1 and 5 have fully merged already, and those in groups 2 and 4 have started to merge, although we still have separate peaks there. Gaps between groups are quite pronounced at this point. Increasing  $w$  to 1.5, the subbands in groups 2 and 4 have also fully merged; and they have also started to merge with the central group (consisting of only 1 subband). At this point the situation has become identical to that of  $q/p = 1/5$ : we have 4 critical energies each carrying TCN +1 away from the band center, while the current carrying states at band center has TCN -4. Further increasing  $w$  to 2.5, we find the current carrying states from the central group have split into two critical energies, and they in turn have already merged with the current carrying states from groups 2 and 4; there remain 4 critical energies as indicated by the 4 peaks in  $\rho_c$ ; 2 of them carry TCN +1, from the edge groups; the other 2 closer to the band center carry TCN -1. All the remaining critical energies have merged and disappeared as  $w$  is increased to 3.5.

### B. Cases with next nearest neighbor hopping

In this subsection we present data for cases where there is next nearest hopping, with the magnitude of the hopping matrix element  $t'$ , in unit of the nearest neighbor hopping. In the presence of  $t'$ , the particle-hole symmetry present in the model with nearest neighbor hopping only is broken. Although Eq. (3) is still satisfied in this case, generically the magnitude of  $\sigma_t$  is not necessarily minimized (by choosing the appropriate  $s$ ). However if  $t'$

is small enough, and if the subband gaps are not closed as  $t'$  is increased from 0, the Chern number of each subband remains the same as that of  $t' = 0$ .

$q/p = 1/3, t' = 0.5$ . In this case we have 3 subbands. The Hall conductance (Chern number) of each subband is +1, +1, and -2. This is *different* from the case with  $t' = 0$ , where the Chern numbers are +1, -2, +1. We find the next nearest neighbor hopping term breaks the particle-hole symmetry, and moves the negative Chern numbers to higher subbands. In Fig. 5 we show how  $\rho$  and  $\rho_c$  for systems of size  $9 \times 9$  evolve with  $w$ . For weak randomness  $w = 0.5$ , the gaps separating different subbands are still finite. Increasing  $w$  to 2.0, the central and upper subbands have merged, and the current carrying states in both subbands have also merged to form one critical energy represented by a single peak, carrying total Chern number -1. Further increasing  $w$  to 4.0, the lower subband also starts to merge with the other subbands, and the two remaining critical energies carrying Chern numbers +1 and -1 respectively start to move together. When  $w$  is increased to 6.0, all the subbands have merged together completely, and the two critical energies have also merged and disappeared.

$q/p = 1/2, t' = 0.5$ . This is a somewhat special case, because with  $t' = 0$ , the system actually respects time-reversal symmetry (since the nearest neighbor hopping matrix elements can be chosen to be real), and the Hall conductances (and Chern numbers) are zero everywhere. Related to this is the fact that the two subbands are degenerate at two points in the Brillouin zone in this case. In the presence of a nonzero  $t'$ , a gap separating the two subbands is opened up, and the Chern numbers of the lower and upper subbands become 1 and -1 respectively. In Fig. 6 we show results of numerical studies when random potential is added to the system, for systems with size  $8 \times 8$ . Previously Ludwig *et al.*<sup>38</sup> studied this case analytically. We see when  $w = 2.0$ , there is still a gap separating the two subbands, and the current carrying states form two critical energies represented by the peaks in  $\rho_c$ . Increasing  $w$  to 4.0, the two subbands have started to merge, and the two critical energies start to move closer together toward the band center. Further increasing  $w$  to 5.0, the two subbands have merged completely, and the two critical energies have almost merged. Further increasing  $w$

to 7.0, the two critical energies have merged and disappeared, and there are very few current carrying states left in such a finite size system.

#### IV. SUMMARY AND DISCUSSION

In this paper we have studied numerically how current carrying states disappear, and the system undergoes phase transitions from a quantum Hall state to an insulating state as disorder strength increases, in tight binding lattice models with non-interacting electrons. In cases with nearest neighbor hopping only (with particle-hole symmetry), we find the following generic pattern: Subbands away from the center of the band form groups with  $q$  subbands, and the total Chern number of such groups is  $+1$ ; while subbands near the band center form a special group with  $M = \text{mod}(p, 2q)$  subbands. In the presence of relatively weak randomness, groups of critical energies away from the band center merge together and form critical energies carrying Hall conductance (or total Chern number) 1; the current carrying states near the band center form (within our resolution) two critical energies carrying negative Chern numbers. Further increasing randomness strength, the critical energies carrying negative Chern numbers move toward the band edge and merge with the critical energies carrying total Chern number 1, and eventually all critical energies and current carrying states disappear in the thermodynamic limit, and the system becomes insulating no matter where the Fermi energy is. Adding a next nearest neighbor hopping term breaks the particle-hole symmetry. We find this tends to push subbands carrying negative Chern numbers to higher energies. However as long as the Fermi energy is near the bottom of the band, the way critical energies carrying positive Chern numbers are killed is similar: they are killed by critical energies carrying negative Chern numbers coming down from high energy.<sup>39</sup>

Our results suggest that the non-trivial structure in the dependence of Hall conductance on electron density in the presence of a periodic potential is quite vulnerable against randomness; in order to see such structures experimentally,<sup>40</sup> one must control the amount of randomness carefully so that subbands in the same group (which are separated by very small

gaps) are not merged together by randomness. As the randomness becomes strong enough to merge subbands in the same group, there remain only critical energies carrying Hall conductance 1 except near the band center. Thus the situation is essentially the same as without periodic potential, i.e., there is no oscillation in the dependence of the Hall conductance on electron density.

In the one-band tight binding model studied here and elsewhere, all current carrying extended states disappear at finite randomness strength for a fixed strength of magnetic field. Based on this observation, Xie *et al.*<sup>24</sup> proposed that in the *continuum* system, instead of floating to infinite energy in the limit  $B \rightarrow 0$ , all extended states disappear at *finite*  $B$  for a given disorder strength. In the following we show that extrapolating the lattice model to the continuum limit is quite subtle, and one needs to be extremely careful in interpreting the implications of results of lattice model on the continuum system. For simplicity we consider the case  $q = 1$ ; extension to cases with  $q > 1$  in the following discussion is straightforward.<sup>41</sup>

Without randomness, there are two energy scales in the lattice model for a given strength of magnetic field, namely, the bandwidth which is of order  $t$  (the hopping matrix element), and the Landau level spacing  $\hbar\omega_c = \hbar e B / m^* c \propto t/p$ , where  $m^*$  is the effective mass at the bottom of the band. There are also two length scales, the lattice spacing  $a$  and the magnetic length  $\ell = \sqrt{\hbar c / e B} \propto a\sqrt{p}$ . In the continuum system however, there is only one energy scale, namely the Landau level spacing  $\hbar\omega_c$ , and one length scale, the magnetic length  $\ell$ ; the band width is infinite, and the lattice spacing is zero. Thus to reach the continuum limit in the tight binding lattice model, one must send both  $t$  and  $p$  to infinity, with the ratio  $t/p$  remain a constant; in the mean time  $a$  should be sent to zero in such a way that  $a\sqrt{p}$  remains a constant.

The random potential introduces another energy scale.<sup>42</sup> Naively, the new scale is  $w$ ; what has been shown in lattice models is that the critical  $w_c$  above which no extended states exist is finite *in units of*  $t$ ,<sup>23-25</sup> and  $w_c$  goes to zero as  $p$  goes to infinity, again *in units of*  $t$ . In the continuum model however, the real new energy scale introduced by the random potential is the amount of broadening of a Landau level,  $\delta E$ . The important question in

the continuum model is *in units of*  $\hbar\omega_c$ , how much  $\delta E$  does one need to kill all the current carrying extended states, or what is the ratio  $\delta E_c/\hbar\omega_c$ ? If the answer is finite, then for a given disorder strength, one needs a finite critical magnetic field in the continuum for extended states to exist, as suggested in Refs. 23,24; while if the answer is infinite, then the extended state must exist no matter how small  $B$  is, and float to infinite energy only in the limit  $B \rightarrow 0$ , as originally proposed by Khmelnitskii<sup>10</sup> and Laughlin.<sup>11</sup> In the lattice model with large  $p$ , we have  $\delta E \propto w/\sqrt{p}$ . This is because the linear size of the wave function in a given Landau level is  $\ell$ , which means the number of lattice sites it occupies is proportional to  $\ell^2 \propto p$ ; since the random potential on different sites are uncorrelated, the shift on the energy is typically of order  $w/\sqrt{p}$ , the magnitude of the *averaged* potential in that area. Therefore we find

$$\frac{\delta E_c}{\hbar\omega_c} \propto \frac{w_c/\sqrt{p}}{t/p} = \frac{w_c\sqrt{p}}{t}. \quad (8)$$

Let us assume

$$w_c \propto 1/p^\alpha \quad (9)$$

in the limit  $p \rightarrow \infty$ . Since  $\delta E$  must be at least comparable to  $\hbar\omega_c$  for current carrying states to disappear, from Eq. (8) we must have  $\alpha \leq 1/2$ . If  $\alpha = 1/2$ , the ratio in Eq. (8) is finite as  $p \rightarrow \infty$ ; while if  $\alpha < 1/2$ , or  $\alpha = 1/2$  but there is logarithmic corrections to Eq. (9), the ratio in Eq. (8) becomes infinite as  $p \rightarrow \infty$ ; the situation becomes qualitatively different. It was shown in Ref. 25 that the data for  $16 \leq p \leq 384$  may be fit reasonably well for  $\alpha = 1/2$ . Since the crucial question is whether  $\alpha$  is *exactly*  $1/2$  or not, we feel it is probably difficult to resolve this issue numerically, and analytic input is needed here.

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<sup>41</sup> We thank S. L. Sondhi for extensive discussions on the following points.

<sup>42</sup> For the uncorrelated random onsite potential studied here (which maps onto Gaussian white noise potential with zero correlation length in the continuum limit), no new length scale is introduced, although it would be interesting to study random potential with finite correlation length as well.

## FIGURES

FIG. 1. Total density of states per unit area ( $\rho$ , left panels), and density of current carrying states ( $\rho_c$ , right panels), for systems with  $3/7$  flux quantum per plaquette, size  $7 \times 7$ , and different disorder strengths ( $w$ ).

FIG. 2. Same as in Figure 1, for systems with  $2/7$  flux quantum per plaquette, size  $7 \times 7$ .

FIG. 3. Same as in Figure 1, for systems with  $3/8$  flux quantum per plaquette, size  $8 \times 8$ .

FIG. 4. Same as in Figure 1, for systems with  $2/9$  flux quantum per plaquette, size  $9 \times 9$ .

FIG. 5. Same as in Figure 1, for systems with  $1/3$  flux quantum per plaquette, magnitude of next nearest neighbor hopping  $t' = 0.5$ , size  $9 \times 9$ .

FIG. 6. Same as in Figure 1, for systems with  $1/2$  flux quantum per plaquette, magnitude of next nearest neighbor hopping  $t' = 0.5$ , size  $8 \times 8$ .

Figure 1

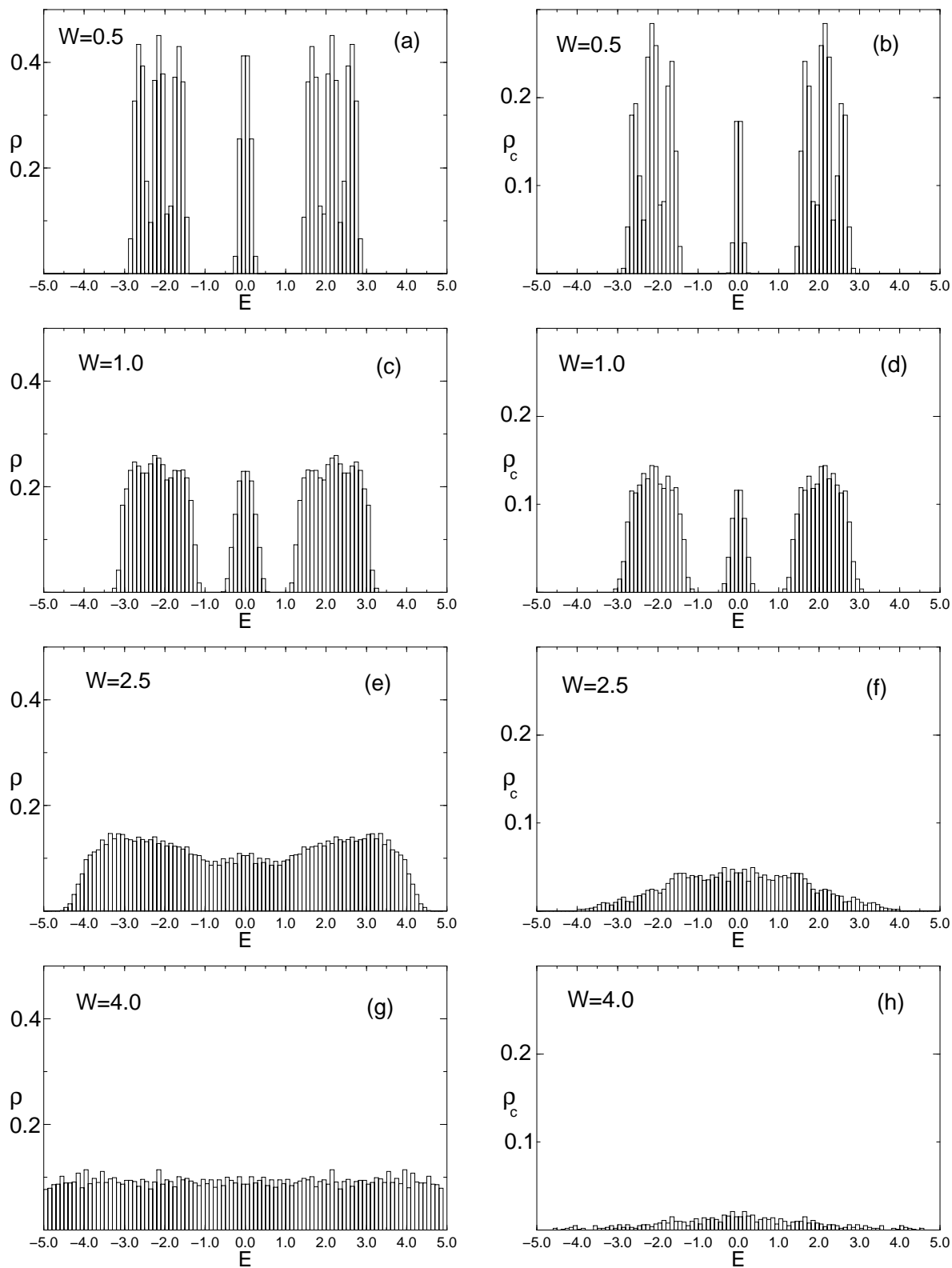


Figure 2

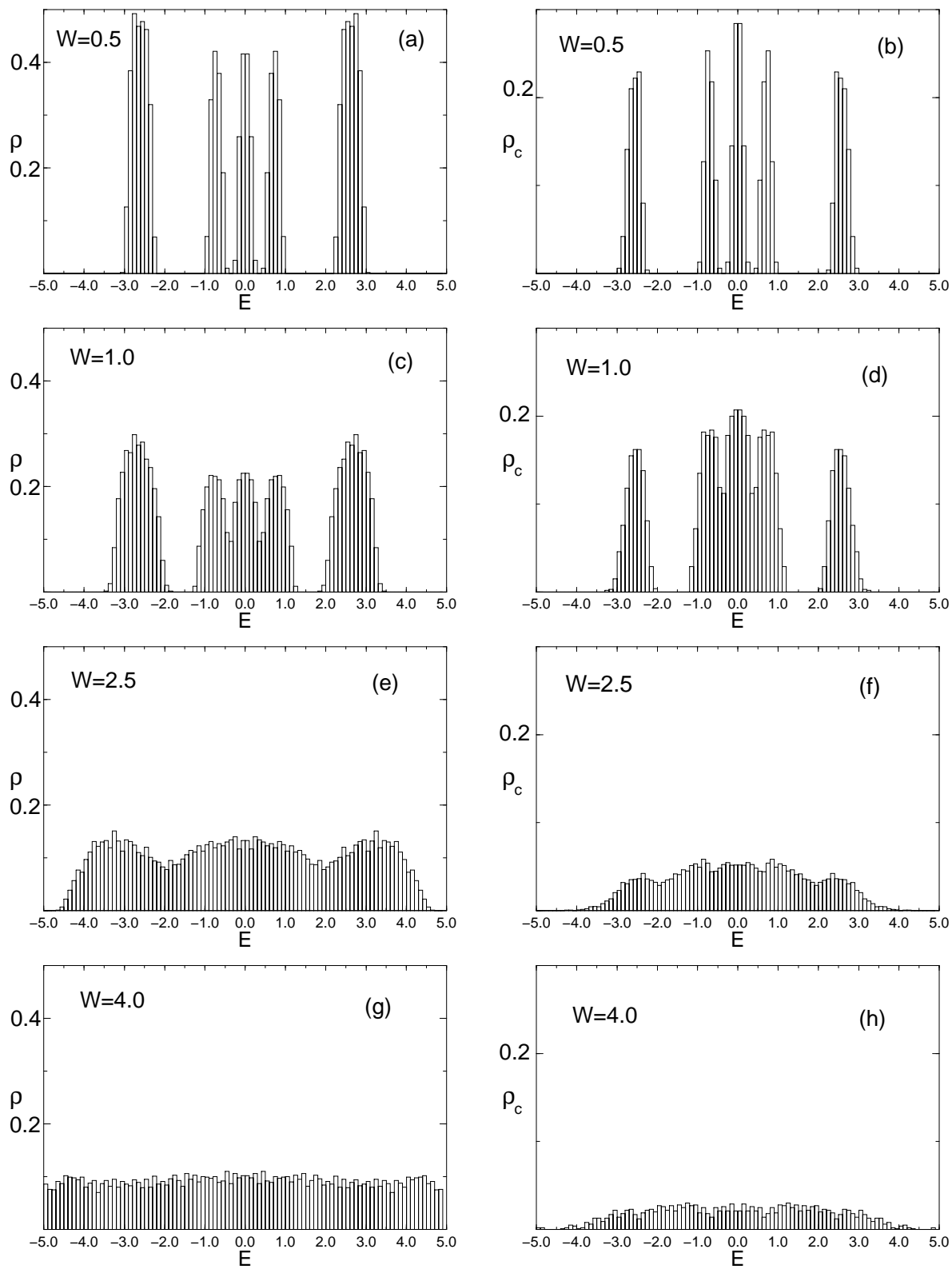


Figure 3

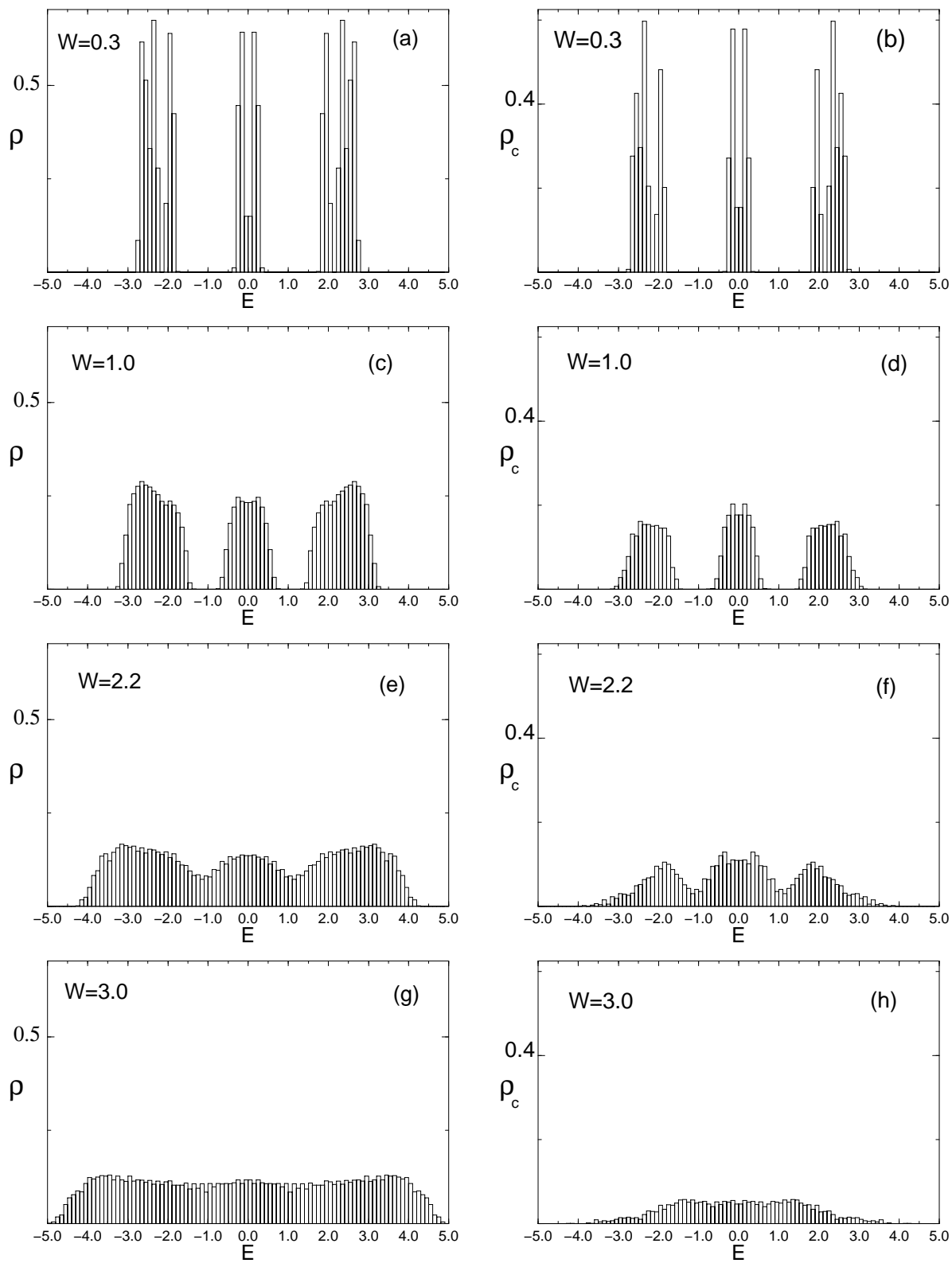


Figure 4

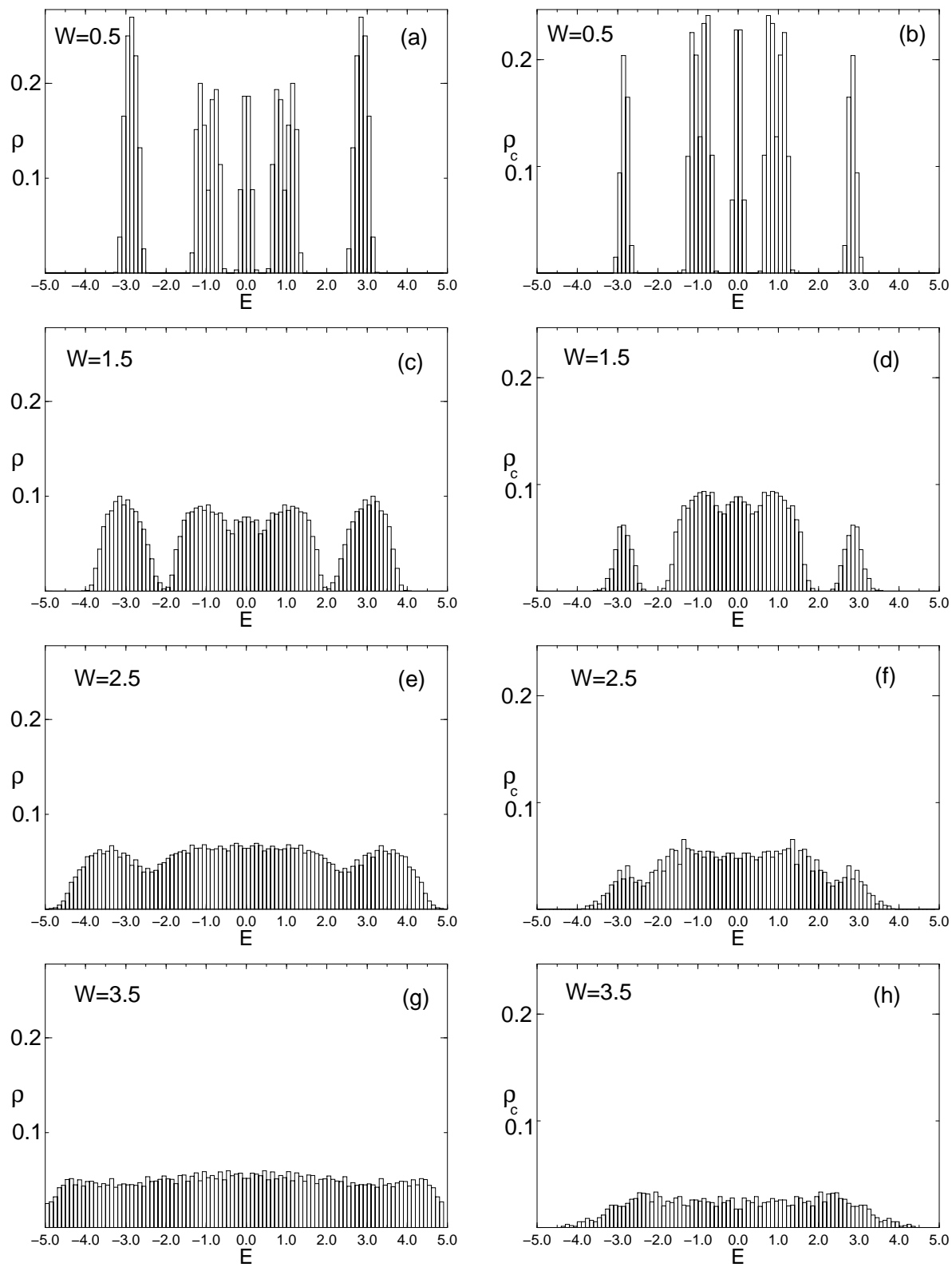


Figure 5

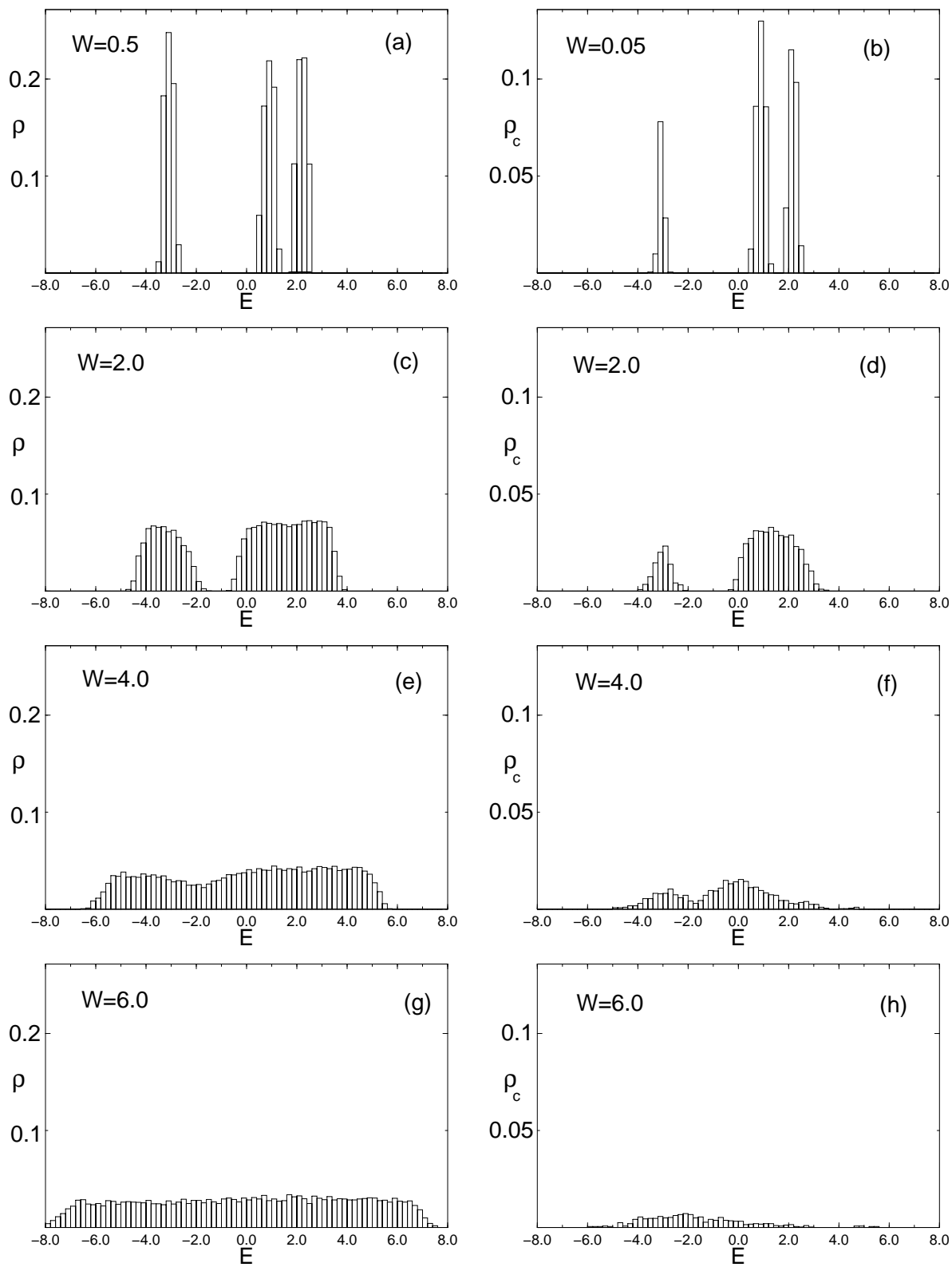


Figure 6

